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LINEAR-TIME SEPARATION ALGORITHMS FOR THE THREE-INDEX ASSIGNMENT POLYTOPE

Egon Balas
Carnegie Mellon University

and

Liqun Qi
University of New South Wales

Carnegie Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

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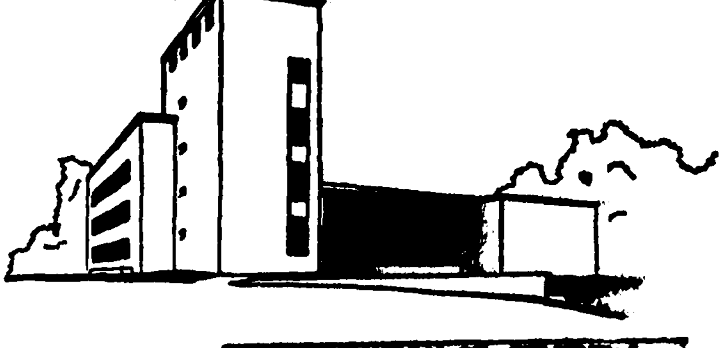

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Egon Balas
Carnegie Mellon University

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University of New South Wales

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Management Science Research Group
Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

Balas and Saltzman [1] identified several classes of facet inducing inequalities for the three-index assignment polytope, and gave $O(n^4)$ separation algorithms for two of them. We give $O(n^3)$ separation algorithms for these two classes of facets, and also for a third class. Since the three-index assignment problem has n^3 variables, these algorithms are linear-time and their complexity is best possible.

Keywords:

Three-index assignment
Facets
Cliques
Odd holes

1. Introduction

Consider three disjoint n-sets and the collection of all weighted triplets with one element in each n-set. The three-index assignment (or three-dimensional matching) problem asks for a minimum-weight set of triplets that partitions the union of the three n-sets. It can be stated as the following 0-1 programming problem:

$$\begin{aligned} &\min \sum \{c_{ijk}x_{ijk} : i \in I, j \in J, k \in K\}, \\ &\text{s.t.} \sum \{x_{ijk} : j \in J, k \in K\} = 1, \ \forall i \in I, \\ &\sum \{x_{ijk} : i \in I, k \in K\} = 1, \ \forall j \in J, \\ &\sum \{x_{ijk} : i \in I, j \in J\} = 1, \ \forall k \in K, \\ &x_{ijk} \in \{0,1\}, \quad \forall i, j, k, \end{aligned} \tag{1.1}$$

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$. Let A be the coefficient matrix of the constraint set of (1.1). Then $R = I \cup J \cup K$ is the row index set of A. Let S be the column index set of A. Let G_A be the intersection graph of A. Then S is the node set of G_A. Let

$$P = \{x \in \Re^{n^3} : Ax = e, x \geq 0\},$$

where $e = (1, ..., 1)^T \in \Re^{3n}$. Then

$$P_I = \text{conv}\{x \in \{0,1\}^{n^3} : x \in P\}$$

is the three-index assignment polytope of order n.

Balas and Saltzman [1] described the cliques of G_A as belonging to three distinct classes and showed that cliques in two of the three classes induce inequalities that define facets of P_I. Furthermore, they gave an $O(n^4)$ procedure for finding a facet-defining clique-inequality violated by a given noninteger $x \in P$, or showing that no such inequality exists. They also described in [1] the odd holes of G_A and identified two classes of facets associated with odd holes that are easy to generate. One class has coefficients of 0 and 1, the other class coefficients of 0, 1 and 2.

In Section 2 of this paper, we give two $O(n^3)$ separation algorithms for the above mentioned two classes of facet defining clique-inequalities. In Section 3, we give an $O(n^3)$ separation algorithm for facet inducing inequalities attributed with lifted odd holes of length 5, having coefficients equal to 0 or 1. Since the number of variables of the three-index assignment problem is n^3, the complexity of these procedures is of the lowest possible order. Our procedures can be used to speed up the performance of the branch and bound algorithm

for the three-index assignment problem developed in [2], which uses facet inducing inequalities as ingredients of a Lagrangian relaxation to obtain strong lower bounds. Our procedures can be extended to more general cases. In Section 4, we present an $O(n^{h+1})$ procedure for detecting a violated facet-defining inequality of P_I, which is associated with a $2h+1$-hole of G_A and belongs to a subclass of the first odd-hole-associated class of facets of [1].

We use the same notation as [1]. Especially, a^s means a column of A associated with $s \in S$. We may specify $s = \{i, j, k\}$ for $i \in I$, $j \in J$, $k \in K$. For a set $Q \subseteq R$, we use Q_I, Q_J and Q_K to denote the parts of Q in I, J and K respectively.

For literature on the three-index assignment problem, also see [3], ..., [8].

2. Facets Induced by Cliques

For every $s \in S$, let

$$C(s) = \{t \in S : a^s \cdot a^t \geq 2\}.$$

Then the node set $C(s)$ induces a clique of size $3n-2$ in G_A [1, Proposition 2.2]. Such a clique is called a clique of class 2. Furthermore, for $n \geq 3$, the inequality

$$x(C(s)) \equiv \sum \{x_t : t \in C(s)\} \leq 1 \tag{2.1}$$

defines a facet of P_I for every $s \in S$ [1, Theorem 3.3]. Given a noninteger $x \in P$, an $O(n^4)$ procedure was presented in [1, Section 4] to detect whether any inequality induced by a clique of class 2 is violated. We now present an $O(n^3)$ procedure to do this.

Algorithm 2.1. Suppose that x is a noninteger point in P. Let ν be an integer greater than or equal to 4.

Step 1. Let $d_s = 0$ for all $s \in S$.

Step 2. Check x_t for all $t \in S$. If

$$x_t \geq \frac{1}{\nu n}, \tag{2.2}$$

then set $d_s := d_s + x_t$ for all $s \in C(t)$. If $d_s > 1$, then stop: (2.1) is violated by x with this s. Otherwise, continue.

Step 3. For $s \in S$, if

$$d_s > \frac{\nu - 3}{\nu}, \tag{2.3}$$

then check whether the inequality (2.1) associated with s is violated and if so, then stop. Else continue. ∎

Theorem 2.2. *Algorithm 2.1 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 2. The value of ν which minimizes the complexity of the algorithm is* 6.

To prove this theorem we need two lemmas. The first lemma is essentially Lemma 4.1 of [1]. We repeat it here for convenience:

Lemma 2.3. *For any $x \in P$ and any positive number λ, the number of components with value $\geq \lambda$ is $\leq n/\lambda$.* ∎

Lemma 4.1 of [1] states this for $1/\lambda$ integer, but the result holds for reals as well. The proof in [1] is somewhat indirect. In fact, the sum of components of $x \in P$ is always n, and all the components of x are nonnegative. The conclusion thus follows.

Lemma 2.4. *For any $x \in P$ and any positive number λ, the number of $s \in S$ such that $x(C(s)) \geq \lambda$ is $\leq n(3n-2)/\lambda$.*

Proof: Consider

$$\sum\{x(C(s)) : s \in S\}.$$

For each $t \in S$, the number of distinct s such that $t \in C(s)$ is $3n-2$. Thus,

$$\sum\{x(C(s)) : s \in S\} = (3n-2)\sum\{x_t : t \in S\} = (3n-2)n.$$

Since $x(C(s)) \geq 0$ for each s, the conclusion follows. ∎

Proof of Theorem 2.2: We first show that Algorithm 2.1 works. By Proposition 2.2 of [1], $|C(s)| = 3n-2$ for every $s \in S$. Suppose that (2.1) is violated for an $s \in S$. Then

$$d_s := \sum\{x_t : t \in C(s), x_t \geq \frac{1}{\nu n}\}$$

$$> 1 - \sum\{x_t : t \in C(s), x_t < \frac{1}{\nu n}\}$$

$$\geq 1 - \frac{3n-2}{\nu n} \geq \frac{\nu - 3}{\nu},$$

i.e., (2.3) holds and the violation is discovered in Step 3. Therefore, Algorithm 2.1 determines whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 2.

We now consider the complexity of the algorithm. The complexity of Step 1 is n^3. According to Lemma 2.3, there are at most νn^2 components of x satisfying (2.2). For each t, the number of s in $C(t)$ is $3n-2$. Hence, the complexity of Step 2 is $\nu n^2(3n-2)$. Finally, if (2.3) holds, then

$$x(C(s)) \geq d_s \equiv \sum\{x_t : t \in C(s), x_t \geq \frac{1}{\nu n}\} > \frac{\nu - 3}{\nu}.$$

By Lemma 2.4, the number of s such that (2.3) holds is not greater than $\frac{\nu n(3n-2)}{\nu-3}$. The number of basic operations to check (2.1) is $3n-2$. We also need to use n^3 basic steps to check (2.3). Therefore, the complexity of Step 3 is $n^3 + \frac{\nu n(3n-2)^2}{\nu-3}$. Hence, the overall complexity of Algorithm 2.1 is $O(n^3)$. The value of ν that minimizes the maximum of the two polynomials expressing the complexity of Steps 2 and 3 is easily seen to be 6. ∎

We now consider facets induced by cliques of class 3 [1]. Suppose that $s, t \in S$ such that

$$a^s \cdot a^t = 0. \tag{2.4}$$

Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$, and

$$C(s,t) = \{s, t_1, t_2, t_3\}.$$

Then the node set $C(s,t)$ induces a (4-)clique in G_A [1, Proposition 2.3]. Such a clique is called a clique of class 3. The number of cliques of class 3 is $\frac{1}{4}n^3(n-1)^3$ [1, Corollary 2.5]. Furthermore, for $n \geq 4$, the inequality

$$x(C(s,t)) \equiv \sum\{x_t : t \in C(s,t)\} \leq 1 \tag{2.5}$$

defines a facet of P_I for every $s, t \in S$ satisfying (2.4) [1, Theorem 3.7]. Given a noninteger $x \in P$, an $O(n^4)$ procedure was also presented in [1, Section 4] to detect whether any inequality induced by a clique of class 3 is violated. We now present an $O(n^3)$ procedure to do this.

Algorithm 2.5. Suppose that x is a noninteger point in P. Check x_s for $s \in S$. If

$$\frac{1}{4} < x_s < 1, \tag{2.6}$$

then check x_p for $p \in S$ satisfying

$$a^s \cdot a^p = 1. \tag{2.7}$$

If

$$x_p > \frac{1 - x_s}{3}, \tag{2.8}$$

then for $t \in S$ such that $a^t \cdot a^s = 0$ and

$$a^t \cdot a^p = 2, \tag{2.9}$$

check whether (2.5) is violated or not. ∎

Theorem 2.6. *Algorithm 2.5 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 3.*

Proof: We first show that Algorithm 2.5 works. Let $C(s,t)$ be the node set of a clique of class 3. Since $|C(s,t)| = 4$, if $x \in P$ violates (2.5), then x has at least one component $> \frac{1}{4}$. By Proposition 2.4 of [1], there is no loss of generality in assuming that this happens for the components indexed by s, i.e., that $x_s > \frac{1}{4}$. Since $x \in P$, we have

$$\sum\{x_{i_s jk} : j \in J, k \in K\} = 1, \tag{2.10}$$

$$\sum\{x_{ij_s k} : i \in I, k \in K\} = 1, \tag{2.11}$$

$$\sum\{x_{ijk_s} : i \in I, j \in J\} = 1. \tag{2.12}$$

If $x_s = 1$, then $x_p = 0$ for any p such that $a^p \cdot a^s = 1$, by (2.10), (2.11) and (2.12). In this case, for any t such that $a^t \cdot a^s = 0$, the left hand side of (2.5) is 1. Thus, if $x \in P$ violates (2.5), $x_s < 1$. This justifies (2.6). Since $|C(s,t)| = 4$, if (2.5) is violated, then (2.8) must hold for at least one $p \in C(s,t)$, $p \neq s$. Therefore, this algorithm determines whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 3.

We now consider the complexity issue. The complexity to check all s is $O(n^3)$. According to Lemma 2.3, there are at most $4n$ components of x satisfying (2.6). For each such component x_s, there are $(n-1)^2$ components x_p satisfying (2.7). So the complexity to find all pairs $s, p \in S$ satisfying (2.6) and (2.7) is $O(n^3)$. By (2.10), (2.11) and (2.12), for given x_s satisfying (2.6), there are at most 6 p such that (2.7) and (2.8) hold. When s and p are fixed and (2.7) is satisfied, there are $n-1$ t satisfying (2.4) and (2.9). One needs 3 additions and one comparison to check (2.5). Therefore, the complexity of the remaining part of the algorithm is $O(n^2)$. Hence, the overall complexity of Algorithm 2.5 is $O(n^3)$. ∎

In the proof, we see that the main work is to check all $s \in S$. This work cannot be reduced. So this order cannot be further improved. Similarly, the order in Theorem 2.2 is also the minimum.

By [1], only these two classes of cliques induce facets of P_I. Combining Theorems 2.2 and 2.6, we have

Theorem 2.7. *One can determine in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of G_A.* ∎

3. Facets Associated with Five-Holes

Assume that $n \geq 3$. Let $Q \subseteq R$, $|Q| = 2h + 1$ for some positive integer h with $h \leq n - 1, 1 \leq |Q_L| \leq h$, $L = I, J, K$, and let

$$S(Q) := \{s \in S : \sum\{a_q^s : q \in Q\} \geq 2\}. \tag{3.1}$$

Then the inequality

$$\sum\{x_s : s \in S(Q)\} \leq h \tag{3.2}$$

defines a facet of P_I [1, Theorem 6.1]. These facets can be regarded as lifted from odd hole inequalities. The number of distinct inequalities (3.2) is $O(2^{3n})$ [1, Proposition 6.5]. In particular, facet defining inequalities (2.1) are special cases of (3.2) with $h = 1$.

Denote the set of Q satisfying the above requirements with a fixed h by S^h. Let $\Phi_h = \{S(Q) : Q \in S^h\}$, where $S(Q)$ is defined by (3.1). Thus, Algorithm 2.1 is an $O(n^3)$ algorithm to detect violated facets induced by sets in Φ_1. In this section, we present an $\mathcal{O}(n^3)$ procedure to detect whether there exists a facet induced by a set in Φ_2, which is violated by a given noninteger $x \in P$. Notice that the number of facets defined by (3.2) with $h = 2$ is $O(n^5)$. Without loss of generality, we may assume that such an x does not violate any inequality (2.1), i.e., assume that Algorithm 2.1 was applied to x first.

Suppose that $x \in P$. For $i \in I, j \in J, k \in K$, we will denote

$$x(i, j, K) := \sum\{x_{ijk'} : k' \in K\}, \tag{3.3}$$

$$x(i, J, k) := \sum\{x_{ij'k} : j' \in J\}, \tag{3.4}$$

$$x(I, j, k) := \sum\{x_{i'jk} : i' \in I\} \tag{3.5}$$

In the following, we establish some formulas to calculate $x(S(Q))$ for $Q \in S^2$. We consider two cases.

First, suppose that $s, t \in S$ satisfy

$$a^s \cdot a^t = 1.$$

Then

$$Q \equiv s \cup t \in S^2.$$

Assume that $s = (i_s, j_s, k_s), t = (i_t, j_t, k_t)$. It is not difficult to see that if $k_s = k_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(i_s, j_t, K) + x(i_t, j_s, K) - x_{i_s j_t k_s} - x_{i_t j_s k_t}; \tag{3.6}$$

if $j_s = j_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(i_s, J, k_t) + x(i_t, J, k_s) - x_{i_s j_t k_t} - x_{i_t j_s k_s}; \tag{3.7}$$

and if $i_s = i_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(I, j_s, k_t) + x(I, j_t, k_s) - x_{i_s j_s k_t} - x_{i_t j_t k_s}. \tag{3.8}$$

Next, suppose that $s, t \in S$ satisfy

$$a^s \cdot a^t = 2. \tag{3.9}$$

Again assume that $s = (i_s, j_s, k_s), t = (i_t, j_t, k_t)$. Then define for the ordered pair (s, t),

$$x(s,t) := x(i_t, J, k_t) + x(i_t, j_t, K) - 2x_{i_t, j_t, k_t}, \tag{3.10}$$

if $i_s \neq i_t$;

$$x(s,t) := x(i_t, j_t, K) + x(I, j_t, k_t) - 2x_{i_t, j_t, k_t}, \tag{3.11}$$

if $j_s \neq j_t$;

$$x(s,t) := x(I, j_t, k_t) + x(i_t, J, k_t) - 2x_{i_t, j_t, k_t}, \tag{3.12}$$

if $k_s \neq k_t$. Notice that in (3.10) $j_t = j_s$ and $k_t = k_s$, in (3.11) $i_t = i_s$ and $k_t = k_s$, and in (3.12) $i_t = i_s$ and $j_t = j_s$. Thus, (3.10) can be written with j_s and k_s substituted for j_t and k_t, and there are similar expressions for (3.11) and (3.12).

In the case that (3.9) holds, we may expand $s \cup t$ to a set $Q \in S^2$ by adding an adequate element. For example, if $i_s \neq i_t$, we may add $j \neq j_s$ and let

$$Q_I = \{i_s, i_t\},\ Q_J = \{j_s, j\},\ Q_K = \{k_s\}, \tag{3.13}$$

or we may add $k \neq k_s$ and let

$$Q_I = \{i_s, i_t\},\ Q_J = \{j_s\},\ Q_K = \{k_s, k\}. \tag{3.14}$$

Notice that there are $2(n-1)$ ways to expand a given pair of s and t satisfying (3.9) to a set $Q \in S^2$. We may also calculate $x(S(Q))$ by expressions similar to (3.6)-(3.8). For example, in case (3.13), one can check that

$$x(S(Q)) = x(C(s)) + x(s,t) + x(i_s, j, K) + x(i_t, j, K) + x(I, j, k_s) - 2x_{i_s j k_s} - 2x_{i_t j k_s}, \tag{3.15}$$

where $x(s,t)$ is given by (3.10). Note that if $i_s \neq i_t$, there are two possibilities, namely, (3.13) and (3.14). Therefore, there are six cases totally, i.e., two cases for $i_s \neq i_t$, two cases for $j_s \neq j_t$, and two cases for $k_s \neq k_t$. In each of these six cases, there is an expression of type (3.15) to calculate $x(S(Q))$. Later, we will refer to these as "expressions of type (3.15)".

We are now ready to describe our algorithm:

Algorithm 3.1. Suppose that x is a noninteger point in P. Suppose (2.1) holds for all $s \in S$.

Step 1. Let $d_s = 0$ for all $s \in S$.

Step 2. Check x_t for all $t \in S$. If

$$x_t \geq \frac{1}{12n}, \tag{3.16}$$

then let $d_s := d_s + x_t$ for all $s \in S$ satisfying

$$a_s \cdot a_t \geq 2.$$

Step 3. For all $s \in S$, if

$$d_s \geq \frac{1}{12}, \tag{3.17}$$

then set $d_s := x(C(s))$.

Step 4. For all $i \in I, j \in J, k \in K$, calculate $x(i,j,K), x(i,J,k)$ and $x(I,j,k)$ by (3.3), (3.4) and (3.5).

Step 5. For all $i \in I, j \in J, k \in K$, form the sets

$$L(i,J,K) := \{s \in S : d_s > \frac{1}{3}, i_s = i\},$$

$$L(I,j,K) := \{s \in S : d_s > \frac{1}{3}, j_s = j\}$$

and

$$L(I,J,k) := \{s \in S : d_s > \frac{1}{3}, k_s = k\}$$

Store them.

Step 6. For each $s = (i_s, j_s, k_s) \in S$ such that $d_s > \frac{1}{2}$, check d_t for each $t \in L(s)$, where

$$L(s) := (L(i_s, J, K) \cup L(I, j_s, K) \cup L(I, J, k_s)) \setminus \{s\}. \tag{3.18}$$

If $a^t \cdot a^s = 1$ and $d_t > (2 - d_s)/3$, calculate $x(S(Q))$ by the appropriate expression (3.6), (3.7) or (3.8), with d_p substituted for $x(C(p)), p = s, t$.

If $a^t \cdot a^s = 2$ and $x(s,t) > (2 - d_s)/3$, calculate $x(S(Q))$ by the appropriate expression of type (3.15), with d_p substituted for $x(C(p)), p = s, t$, for every $Q \in S^2$ such that $s \cup t \subseteq Q$.

If $x(S(Q)) > 2$ for some Q, stop: the corresponding inequality (3.2) is violated by x. Otherwise, continue. ∎

Theorem 3.2. *Algorithm 3.1 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality (3.2) with $h = 2$.*

We prove several auxiliary results.

Lemma 3.3. *After Step 3 of Algorithm 3.1, for all $s \in S$,*

$$d_s \leq x(C(s)). \tag{3.19}$$

If $x(C(s)) > \frac{1}{3}$, then $d_s = x(C(s))$.

Proof: It is obvious that (3.19) holds. Suppose that $x(C(s)) > \frac{1}{3}$. Then

$$d_s = \sum\{x_t : t \in C(s), x_t \geq \frac{1}{12n}\} > \frac{1}{3} - \sum\{x_t : t \in C(s), x_t < \frac{1}{12n}\}$$

$$\geq \frac{1}{3} - \frac{3n-2}{12n} \geq \frac{1}{12},$$

i.e., $d_s \geq \frac{1}{12}$ holds and in this case d_s is set equal to $x(C(t)$ in Step 3. ∎

Lemma 3.4. *At Step 6,*

$$|L(s)| \leq 27n, \tag{3.20}$$

$$|\{t \in L(s) : a^s \cdot a^t = 2, x(s,t) > \frac{1}{3}\}| \leq 18, \tag{3.21}$$

where $L(s)$ is defined by (3.18) and $x(s,t)$ is defined by (3.10), (3.11) and (3.12).

Proof: Suppose that $s = (i_s, j_s, k_s)$.

On the one hand, by Lemma 3.3 and the definition of $L(i_s, J, K)$ at Step 5,

$$\sum\{x(C(t)) : t \in L(i_s, J, K)\} \geq \frac{1}{3}|L(i_s, J, K)|.$$

On the other hand,

$$\sum\{x(C(t)) : t \in L(i_s, J, K)\}$$

$$= \sum_{j_t, k_t}\{x(C(t) : t = (i_s, j_t, k_t)\}$$

$$\leq \sum_{j_t, k_t}[\sum_i x_{ij_tk_t} + \sum_j x_{i_sjk_t} + \sum_k x_{i_sj_tk}]$$

$$= \sum_i \sum_{j_t, k_t} x_{ij_tk_t} + \sum_{j_t}\sum_{j,k_t} x_{i_sjk_t} + \sum_{k_t}\sum_{j_t,k} x_{i_sj_tk}]$$

$$= \sum_{i} 1 + \sum_{j_t} 1 + \sum_{k_t} 1 = 3n.$$

Thus, $|L(i_s, J, K)| \leq 9n$. Similarly, we can show that $|L(I, j_s, K)| \leq 9n$ and $|L(I, j_s, K)| \leq 9n$. From the definition of $L(s)$, we then have (3.20).

It is also seen that

$$\sum \{x(s,t) : t \in S, a^s \cdot a^t = 2\}$$

$$\leq 2 \sum_{j,k} x_{i_s jk} + 2 \sum_{i,k} x_{ij_s k} + 2 \sum_{i,j} x_{ijk_s} = 6.$$

Hence, (3.21) holds. ∎

Lemma 3.5. *Suppose that $Q \in S^2$. Let*

$$H(Q) := \{s \in S : \sum \{a_q^s : q \in Q\} = 3\}.$$

If the inequality (3.2) with $h = 2$ is violated by x, then there are $s, t \in H(Q)$, $s \neq t$ such that

(i). $x(C(s)) > \frac{1}{2}$;

(ii). either $a^s \cdot a^t = 1$ and $x(C(t)) > (2 - x(C(s)))/3$, or $a^s \cdot a^t = 2$ and $x(s,t) > (2 - x(C(s)))/3$.

Proof: Without loss of generality, assume that $Q_I = \{i_1, i_2\}$, $Q_J = \{j_1, j_2\}$, $Q_K = \{k\}$. Then $H(Q) = \{(i_1, j_1, k), (i_1, j_2, k), (i_2, j_1, k), (i_2, j_2, k)\}$, and

$$2 < x(S(Q)) \leq x(C((i_1, j_1, k))) + x(C((i_1, j_2, k))) + x(C((i_2, j_1, k))) + x(C((i_2, j_2, k))).$$

Then at least one term on the right hand side of the above inequality is greater than $\frac{1}{2}$. Without loss of generality, assume it is the first term. Let $s = (i_1, j_1, k)$. Then (i) holds.

Denote $t_1 = (i_2, j_2, k), t_2 = (i_1, j_2, k), t_3 = (i_2, j_1, k)$. Then

$$2 < x(S(Q))$$

$$\leq x(C(s) + x(C(t_1)) + x(i_1, j_2, K) + x(i_2, j_1, K) - x_{i_1 j_2 k} - x_{i_2 j_1 k}, \quad \text{(by (3.6))}$$

$$\leq x(C(s)) + x(C(t_1)) + x(s, t_2) + x(s, t_3). \quad \text{(by (3.11) and (3.10))}$$

Notice that $a^s \cdot a^{t_1}$ and $a^s \cdot a^{t_2} = a^s \cdot a^{t_3} = 2$. Thus,

$$2 - x(C(s)) < x(C(t_1) + x(s, t_2) + x(s, t_3).$$

At least one term at the right hand side of above inequality is greater than one third of the left hand side of this inequality. Hence, (ii) holds. ∎

Proof of Theorem 3.2: By Lemmas 3.5 and 3.3, if the inequality (3.2) with $h = 2$ is violated by x for some Q, there are $s, t \in S$ such that $d_s > \frac{1}{2}$, and either $a^s \cdot a^t = 1$ and $d_t > (2 - d_s)/3$, or $a^s \cdot a^t = 2$ and $x(s,t) > (2 - d_s)/3$.

Thus if $t \in L(s)$, i.e., if $d_t > \frac{1}{3}$, then Step 6 will detect the violation. To see that this is the case. notice that if $a^s \cdot a^t = 1$, then

$$d_t > (2 - d_s)/3 = (2 - x(C(s)))/3 \geq \frac{1}{3},$$

(since $x(C(s)) \leq 1$ for all $s \in S$), and if $a^s \cdot a^t = 2$, then

$$\begin{aligned} d_t &= x(C(t)) \geq x(s,t) \qquad \text{(from the definitions)} \\ &> (2 - d_s)/3 = (2 - x(C(s)))/3 \geq \frac{1}{3}, \end{aligned}$$

Thus in both cases $t \in L(s)$.

Consider now the complexity of Algorithm 3.1. Steps 1 and 2 are obviously $O(n^3)$. In Step 3, the number of $s \in S$ such that $d_s \geq \frac{1}{12}$ is $\leq 12n(3n-2)$ (From Lemma 2.4 and the fact that $d_s \leq x(C(s))$; and for each number s, calculating $x(C(s))$ requires $3n - 3$ additions; thus the complexity of Step 3 is also $O(n^3)$. Further, each of Steps 4 and 5 again requires $O(n^3)$ operations.

Finally, consider the complexity of Step 6. By Lemmas 3.3 and 2.4, the number of s such that $d_s > \frac{1}{2}$ is $O(n^2)$. From Lemma 3.4, the cardinality of $L(s)$ is $O(n)$. Forming $L(s)$ itself is $O(1)$, since its constituent sets have been calculated and stored in Step 5. Thus, checking d_t for each $t \in L(s)$ for all s such that $d_s > \frac{1}{2}$ is $O(n^3)$. If $a^s \cdot a^t = 1$ and $d_t > (2 - d_s)/3$, calculating $x(S(Q))$ is $O(1)$, since the expressions $x(C(s))$, $x(C(t))$ and $x(i,j,K)$, $x(i,J,k)$, $x(I,j,k)$ have been calculated in Step 3 and Step 4 respectively. On the other hand, from Lemma 3.4 the number of those $t \in L(s)$ (for given s) such that $a^s \cdot a^t = 2$ and $x(s,t) > \frac{1}{3}$ is $O(1)$. For each such t, there are $2(n-1)$ ways to expand the pair (s,t) to a set $Q \in S^2$, and for each such set Q, calculating $x(S(Q))$ is again $O(1)$, since the constituent terms of the expressions of type (3.15) needed for this have already been calculated. Thus, the complexity of dealing with all pairs (s,t) satisfying $a^s \cdot a^t = 2$ is again $O(n^3)$, which leaves the total complexity of Step 6 at $O(n^3)$. ∎

4. Facets Associated with Other Odd Holes

It is somewhat more complicated to detect violated facets defined by (3.2) with $h > 2$. We first discuss some subclasses of S^h and Φ_h. Assume that $n \geq 3$. Let $\bar{S}^h$ be the subcollection of S^h such that for any

$Q \in \bar{S}^h$ there exists an $L = I$, J, or K such that $|Q_L| = 1$. Note that in this case $|Q_{L'}| = h$ for $L' \in \{I, J, K\}$, $L' \neq L$. Let $\bar{\Phi}_h = \{S(Q) : Q \in \bar{S}^h\}$. Clearly, $\bar{S}^1 = S^1$, $\bar{S}^2 = S^2$, but $\bar{S}^h$ is a proper subset of S^h for $h > 2$. Assume that $h \geq 3$. We will show that

Theorem 4.1. *For a given $x \in P$, it is possible in $O(n^{h+1})$ steps to detect whether there exists a facet of P_I, which is defined by (3.2) with $Q \in \bar{S}^h$ and violated by x.*

Again, we prove several auxiliary results.

Suppose that $x \in P$ does not violate any inequr'''r

$$x(S(Q)) \leq r. \tag{4.1}$$

with $Q \in \bar{S}^r, r = 1, 2, ..., h-1$, but violates (4.1) with $h = r$ and a set $Q \in \bar{S}^h$. Suppose that h is fixed, not dependent on n.

Lemma 4.2. *Let $H(Q)$ be defined as in Lemma 3.5. Then there exists an $s \in H(Q)$ such that*

$$x(C(s)) > \frac{1}{h}. \tag{4.2}$$

Proof: Clearly,

$$h < x(S(Q)) \leq \sum\{x(C(s)) : s \in H(Q)\}.$$

However, for $Q \in \bar{S}^h$, $|H(Q)| \leq h^2$. The conclusion thus follows. ∎

Since $Q \in \bar{S}^h$, without loss of generality, assume that $Q_K = \{k\}$. Then $|Q_I| = |Q_J| = h$. Suppose now that $s_1 = (i_1, j_1, k) \in H(Q)$ and (4.2) holds for $s = s_1$.

Lemma 4.3. *One of the following claims holds.*

(i). $Q_I = \{i_1, i_2, ..., i_h\}$, where $i_1, i_2, ..., i_h$ are distinct, and $Q_J \supseteq \{j_1, j_2, ..., j_\tau\}$, where $\tau \leq h$, $j_1, j_2, ..., j_\tau$ are distinct, such that

$$x(C(s_\nu)) > \frac{1}{2h-1}, \tag{4.3}$$

for $\nu = 2, ..., h$, where $s_\nu = (i_\nu, j_\nu, k)$, and $j_\nu \in \{j_1, ..., j_\tau\}$ for $\tau < j \leq h$;

(ii). $Q_I \supseteq \{i_1, i_2, ..., i_\tau\}$, where $\tau \leq h$, $i_1, i_2, ..., i_\tau$ are distinct, and $Q_J = \{j_1, j_2, ..., j_h\}$, where $j_1, j_2, ..., j_h$ are distinct, such that (4.3) holds, for $\nu = 2, ..., h$, where $s_\nu = (i_\nu, j_\nu, k)$, and $i_\nu \in \{i_1, ..., i_\tau\}$ for $\tau < i \leq h$.

Proof: Suppose that (4.3) holds for s_ν, $\nu = 2, ..., \tau$ such that there are no repetitions in $\{i_1, ..., i_\tau\}$ and $\{j_1, ..., j_\tau\}$. Suppose that such a collection is maximal, i.e., for any $i \in Q_I \setminus \{i_1, ..., i_\tau\}$ and $j \in Q_J \setminus \{j_1, ..., j_\tau\}$,

$$x(C((i,j,k))) \leq \frac{1}{2h-1}. \tag{4.4}$$

If the conclusion of this lemma does not hold, then there exists $i_0 \in Q_I \setminus \{i_1, ..., i_\tau\}$ and $j_0 \in Q_J \setminus \{j_1, ..., j_\tau\}$ such that

$$x(C((i_0, j_\nu, k))) \leq \frac{1}{2h-1} \tag{4.5}$$

$$x(C((i_\nu, j_0, k))) \leq \frac{1}{2h-1}, \tag{4.6}$$

for $\nu = 1, ..., \tau$. However,

$$x(S(Q)) \leq x(S(Q')) + \sum\{x(C(i_0, j, k) : j \in Q_J,\ j \neq j_0\} + \sum\{x(C(i, j_0, k) : j \in Q_J\}, \tag{4.7}$$

where Q' is defined by $Q'_I = Q_I \setminus \{i_0\}$, $Q'_J = Q_J \setminus \{j_0\}$, $Q'_K = Q_K = \{k\}$. By (4.4), (4.5) and (4.6), each term in the right hand side of (4.7), except the first term, is not greater than $\frac{1}{2h-1}$. The number of such terms is $2h-1$. Clearly, $Q' \in \bar{S}^{h-1}$. Thus, $x(S(Q')) \leq h-1$ by our assumption. Thus, the right hand side of (4.7) is not great than h. This forms a contradiction with our assumptions on x and Q. Therefore, this lemma holds. ■

The following is a brief proof of Theorem 4.1. Cumbersome details are omitted.

Proof of Theorem 4.1: By changing 12 to $4(2h-1)$ in (3.16), and (3.17), we may calculate all $x(C(s)) > \frac{1}{2h-1}$ in $O(n^3)$ steps. Check $s \in S$ with $x(C(s)) \geq \frac{1}{h}$. The number of such s is $O(n^2)$. Let $s_1 = s = (i_1, j_1, k_1)$. We first check $Q \in \bar{S}^h$ such that $s \subseteq Q$ and Q has a structure as described in Lemma 4.3 (i).

Let $k = k_1$. For $\tau = 2, ..., h$, find all possible $s_\nu = (i_\nu, j_\nu, k)$ for $\nu = 2, ..., \tau$ such that (4.3) holds and there are no repetitions in $\{i_1, ..., i_\tau\}$ and $\{j_1, ..., j_\tau\}$. (We may find $s_2, ..., s_\tau$ sequentially.) Similarly to Lemma 3.4, one may see that the number of $t = (i', j', k)$ such that

$$x(C(t)) > \frac{1}{2h-1}$$

is $O(n)$. Thus, the number of such $(s_2, ..., s_\tau)$ is $O(n^{\tau-1})$. Then we find all possible $s_\nu = (i_\nu, j_\nu, k)$ for $\nu = \tau+1, ..., h$ such either (i) or (ii) of Lemma 4.3 holds. With an argument similar to Lemma 3.4, one may see that the number of such s_ν is $O(1)$. In order to expand $s_1 \cup ... \cup s_h$ to a set $Q \in \bar{S}^h$, we need to add it with

$$\{j_{\tau+1}, ..., j_h\} \subseteq J \setminus \{j_1, ..., j_\tau\}.$$

Therefore, the number of expansion ways is $O(n^{h-\tau})$. For each Q produced in this way, we may use formulas simiiar to (3.6)-(3.8), (3.15) to calculate $x(S(Q))$ and check whether

$$x(S(Q)) \leq h. \tag{4.8}$$

The complexity of such formulas are $O(1)$. Summing up the above discussion, one see that the complexity of check (4.8) for all $Q \in \bar{S}^h$ such that $s \subseteq Q$ and Q has a structure as described in Lemma 4.3 (i), for a fixed s, is $O(n^{h-1})$.

The case of Lemma 4.3 (ii) and the cases, where i_1 or j_1 in s is fixed, are similar. Since the number of such s is $O(n^2)$, the overall complexity of such a procedure is $O(n^{h+1})$. This completes the proof of this theorem. ∎

In general, the order of procedures to detect violated facets induced by sets in Φ_h for $h \geq 3$ may be higher than $O(n^{h+1})$. For example, let $h = 4$. We may find $O(n^6)$ triples of (s, t, p) such that $s, t, p \in S$, $x(C(s)), x(C(t)), x(C(p)) > \frac{1}{4}$, and

$$a^s \cdot a^t = 0,\ a^t \cdot a^p = 0,\ a^s \cdot a^p = 0.$$

Then $Q := s \cup t \cup p \in S^4$. We need to check whether

$$x(S(Q)) \leq 4$$

or not. Thus, the complexity of such a procedure is at least $O(n^6)$. But $6 > h+1 \equiv 5$. Hence, the complexity for detecting violated facets induced by sets in Φ_h may be higher than the complexity corresponding to $\bar{\Phi}_h$ for $h \geq 3$.

REFERENCES

[1] E. Balas and M. J. Saltzman, "Facets of the three-index assignment polytope", *Discrete Applied Math.* 23 (1989) 201-229.

[2] E. Balas and M. J. Saltzman, "An algorithm for the three-index assignment problem", to appear in: *Oper. Res.*

[3] A. M. Frieze, "Complexity of a 3-dimensional assignment problem", *European J. Oper. Res.* 13 (1983) 161-164.

[4] A. M. Frieze and J. Yadegar, "An algorithm for solving 3-dimensional assignment problem with application to scheduling a teaching practice", *J. Oper. Res. Soc.* 32 (1981) 989-995.

[5] P. Hansen and L. Kaufman, "A primal-dual algorithm for the three-dimensional assignment problem", *Cahiers du CERO* 15 (1973) 327-336.

[6] O. Leue, "Methoden zur Lösung dreidimensionaler Zuordungsprobleme", *Angew. Inform.* (1972) 154-162.

[7] W. P. Pierskalla, "The tri-substitution method for the three-dimensional assignment problem", *CORS J.* 5 (1967) 71-81.

[8] W. P. Pierskalla, "The multidimensional assignment problem", *Oper. Res.* 16 (1968) 422-431.

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Balas and Saltzman (1) identified several classes of facet inducing inequalities for the three-index assignment polytope, and gove $O(n^3)$ separation algorithms for these two classes of facets, and also for a third class. Since the three-index assignment problem has n^3 variables, these algorithms are linear-time and their complexity is best possible.